Accumulation and bifurcation points of the discontinuous logistic map

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For most values of $x_d \in (-1, 1)$, the logistic map with a sectional discontinuity at $x = x_d$ possesses at least one inverse cascade. By using the property that, when x_d is positive or zero, every first cascade accumulates at a parameter $a = a_{acc}$ immediately at the end of a 2-cycle, we explain the functional dependence of a_{acc} on x_d. Further, we derive hitherto unknown, general analytical expressions for a_{acc} when x_d lies in the range (0, 0.9); in particular, these expressions give values of a_{acc} identical to those previously found by a computational technique for selected values of x_d in the same range [B. L. Tan and T. T Chia, Phys. Rev. E **47,** 3087 (1993)]. We also present a method for calculating the values of the bifurcation points within any inverse cascade for this map and for the TB map which consists of two piecewise linear portions [A. S. Lima, I. C. Moreira, and A. M. Serra, Phys. Lett. A **190,** 403 (1994)].

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I. Introduction

The subject of dynamical systems has been boosted by the study of the behavior of one-dimensional maps in a paper by May [1]. Though simple in form, these maps possess many interesting and surprising properties such as the occurrence of both periodic and chaotic orbits, and universal properties such as the period-doubling route to chaos. The interest in such simple maps has been maintained as they can be models of more complicated dynamical systems that are encountered experimentally.

More recently, there has been much interest in the behavior of one-dimensional discontinuous maps which, as the name suggests, consist of at least two continuous portions separated by gaps. In part of the parameter range, the properties of such a discontinuous map may be related to the properties of one of these continuous portions. Further, the properties should depend on the size of the gaps. Thus, if the gap size approaches zero, the properties of the discontinuous map must approach those of the corresponding continuous map.

The interest in discontinuous maps arises because they have been encountered in experiments on neuronal dynamics [2] and on the diffraction of a laser beam by a hybrid acousto-optic device [3], and also because these maps possess very interesting properties that are not present in continuous maps.

One such property peculiar to discontinuous maps is the occurrence, within a parameter range in the periodic region, of an inverse cascade which consists of a series of stable orbits whose periods form a decreasing arithmetic progression as the parameter increases. Each of these inverse cascades terminates at a definite parameter value known as the accumulation point a_{acc} of the inverse cascade.

For a complete understanding of the model of a dynamical system based on these discontinuous maps we must at least identify the members of these inverse cascades as well as determine the values of their accumulation points.

One map that has been well studied is the logistic map with a sectional discontinuity at $x = x_d = 0$ [5–9]:

$$F(x_n) = \begin{cases} R(x_n) \equiv 1 \quad - a|x_n|^2 & \text{if } x_n > x_d, \\ L(x_n) \equiv 0.9 - a|x_n|^2 & \text{if } x_n \le x_d. \end{cases} \quad (1)$$

In our study of this map with any arbitrary value of x_d, we find that these discontinuous logistic maps with $x_d = 0$ and $x_d \neq 0$ share some common properties such as the occurrence of inverse cascades with the same bifurcation mechanism. Further, the periods of the stable cycles within each inverse cascade are related to one another in very interesting ways [4, 5]. We have also computationally determined the values of the accumulation points a_{acc} of the inverse cascades of the above map F for some selected values of x_d.

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In this paper we shall explain the functional dependence of a_{acc} on x_d. We shall also derive hitherto unknown, general analytical expressions for a_{acc} when x_d lies in the range (0, 0.9); in particular, these expressions give values of a_{acc} identical to those previously found by a computational technique for selected values of x_d in the same range, indicating the correctness of the previous work [4]. Further, by using the property that bifurcations within an inverse cascade occur whenever a cycle element approaches the discontinuity of the map, we shall show that the values of the bifurcation points satisfy polynomial equations which yield solutions that are identical to the computational values. Note that, as the cycle elements of the discontinuous maps are in general basin-dependent, we have chosen the initial value of x to be 0.5 throughout our computations. Finally, as a further illustration of the general usefulness of this technique, we have also applied it to the TB map which consists of two piecewise linear portions [10].

II. Accumulation Points

A) Explanation for variation of a_{acc} with x_d

For convenience, the previously-determined periods of the cycles in the periodic region of the map F with different values of x_d [4] are reproduced in Table 1, which shows that inverse cascades are always present for all values of x_d, except -0.1. In the second column of Table 2 we show the numerical values of a_{acc} of these inverse cascades found previously.

We can see from Table 1 that for positive values of x_d, each first-level inverse cascade, i.e., the most prominent inverse cascade, accumulates at a point $a = a_{acc}$ immediately at the end of a 2-cycle. Further, we can make the following deduction from our computational results: for $a < a_{acc}$, both the 2-cycle elements fall on only one branch of the discontinuous map F when x_d is positive. In particular, when $0 \le x_d < 0.61541$, both cycle elements lie on the right-hand branch R, while for larger values of x_d, they lie on the left-hand branch L and thereby R is "unoccupied". Hence, for $a < a_{acc}$ and a given positive value of x_d, if F is replaced in the whole range by either a continuous map R (if L is unoccupied) or L (if R is unoccupied), then this new continuous map will possess the same 2-cycle elements as the original map F. Using this observation, we shall now explain the variation of the numerical values of a_{acc} with x_d given in Table 2.

Table 1. First-level inverse cascades and other sequences in the periodic region of F for various values of x_d. The symbol 'pd' denotes period-doubling. Chaos occurs immediately after the last cycles shown for all the values of x_d except those labelled by "#", for which there exist other terms that appear to belong to higher-level inverse and direct cascades before the onset of chaos.

x_d	Sequences
−0.3 (#)	$1 \xrightarrow{pd} 2 \xrightarrow{pd} 4, \cdots 50 \to 46 \to 42 \to 38 \to 34$
−0.2 (#)	$1 \xrightarrow{pd} 2,\ 4 \xrightarrow{pd} 8, \cdots 60 \to 52 \to 44 \to 36 \to 28$
−0.1	$1 \xrightarrow{pd} 2,\ 6,\ 4 \xrightarrow{pd} 8,\ 14 \xrightarrow{pd} 28,\ 44$
−0.01	$1 \xrightarrow{pd} 2,\ 8,\ 6,\ 4 \xrightarrow{pd} 8,\ 2 \xrightarrow{pd} 4,$ $\cdots 43 \to 39 \to 35 \to 31 \to 27,\ 31,\ 35,\ 39,\ 43$
0.01	$1 \xrightarrow{pd} 2, \cdots 14 \to 12 \to 10 \to 8 \to 6,\ 6,\ 4 \xrightarrow{pd} 8,\ 2 \xrightarrow{pd} 4$
0.1	$1 \xrightarrow{pd} 2, \cdots 12 \to 10 \to 8 \to 6 \to 4,\ 4,\ 2 \xrightarrow{pd} 4 \xrightarrow{pd} 8,$ $37 \to 103 \to 37 \to 103 \to 37 \to \cdots$
0.2	$1 \xrightarrow{pd} 2, \cdots 10 \to 8 \to 6 \to 4 \to 2 \xrightarrow{pd} 4 \xrightarrow{pd} 8 \xrightarrow{pd}$ $16 \xrightarrow{pd} 32 \xrightarrow{pd} \cdots$
0.3	$1 \xrightarrow{pd} 2, \cdots 10 \to 8 \to 6 \to 4 \to 2 \xrightarrow{pd} 4 \xrightarrow{pd} 8 \xrightarrow{pd}$ $16 \xrightarrow{pd} 32 \xrightarrow{pd} \cdots$
0.4 (#)	$1 \xrightarrow{pd} 2, \cdots 20 \to 18 \to 16 \to 14 \to 12$
0.5	$1 \xrightarrow{pd} 2, \cdots 98 \to 96 \to 94 \to 92 \to 90$
0.6	$1 \xrightarrow{pd} 2, \cdots 175 \to 173 \to 171 \to 169 \to 167$
0.65	$1 \xrightarrow{pd} 2, \cdots 57 \to 55 \to 53 \to 51 \to 49$
0.7	$1 \xrightarrow{pd} 2, \cdots 392 \to 390 \to 388 \to 386 \to 384$
0.8 (#)	$1 \xrightarrow{pd} 2, \cdots 12 \to 10 \to 8 \to 6 \to 4$

Table 2. Accumulation points of the first-level inverse cascades given in Table 1. The numerical values shown are all rounded up to the next nearest five decimal places.

x_d	Accumulation point	
	Numerical value	Analytical value
−0.3	1.30067	
−0.2	1.28897	
−0.01	1.53055	
0.01	0.99020	$50(101-3\sqrt{1133})$
0.1	0.91674	$5(11-3\sqrt{13})$
0.2	0.85787	$5(3-2\sqrt{2})$
0.3	0.81525	$\frac{5}{9}(13-\sqrt{133})$
0.4	0.78465	$\frac{5}{8}(7-\sqrt{33})$
0.5	0.76394	$3-\sqrt{5}$
0.6	0.75237	$\frac{5}{9}(4-\sqrt{7})$
0.65	0.83539	$\frac{10}{169}(31-\sqrt{285})$
0.7	0.84225	$\frac{10}{49}(8-\sqrt{15})$
0.8	0.87934	$\frac{5}{64}(17-\sqrt{33})$

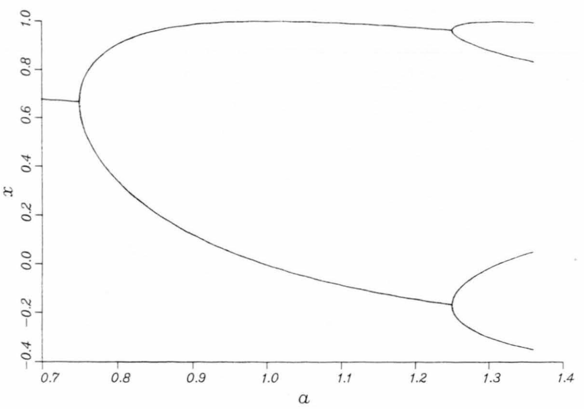

Fig. 1. Bifurcation diagram for the map R' defined by (2).

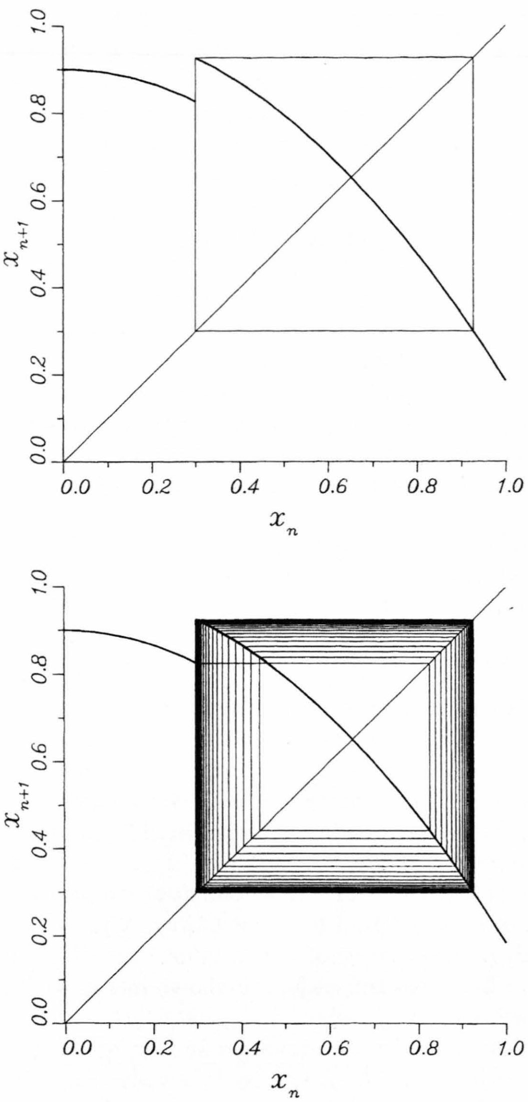

Fig. 2. The map F defined by (1) with $x_d = 0.3$: (a) Parameter $a = 0.81524$, which is slightly smaller than a_{acc}, showing a 2-cycle with one element very close to x_d. (b) Parameter $a = 0.81525$, which is slightly larger than a_{acc}, showing a 68-cycle.

Figure 1 shows a part of the bifurcation diagram of the map defined by

$$R'(x_n) = 1 - a x_n^2, \; x_n \in [-1, 1]. \qquad (2)$$

Note that the equations for R and R' are the same, but defined over different domains. For the discontinuous logistic map F, with a constant value of x_d in the range $[0, 0.61541)$ and parameter $a < a_{acc}$, both the 2-cycle elements fall on the right-hand branch R. Hence, the portion of the bifurcation diagram of F for $a < a_{acc}$ is identical to a portion of that of R' shown in Fig. 1 for the same range of a. Moreover, it is found that if a is increased to just before a_{acc}, then one of the 2-cycle elements of F will approach x_d monotonically. Thus we expect that, when a is smaller than a_{acc} by an infinitesimally small quantity, there will be a 2-cycle element which will be infinitesimally close to x_d. Further, when a is equal to a_{acc}, then, as the map now possesses a RARP property with a truly infinite period, one of its cycle elements must now be located at x_d [11].

As an illustration, Fig. 2a shows the 2-cycle of the map F with $x_d = 0.3$ and $a = 0.81524$, which is just smaller than a_{acc}: here both the 2-cycle elements are located on the right branch R with one of them slightly larger than x_d. When a is increased to 0.81525 which is slightly larger than a_{acc}, the map will possess a cycle with a period of 68 with cycle elements located on both branches, as shown in Figure 2b.

We can now deduce the value of the first accumulation point a_{acc} of the map F when $x_d \in [0, 0.61541)$ from the fact that when a is extremely close to a_{acc}, one of the 2-cycle elements is extremely close to x_d. From the bifurcation diagram of R' shown in Fig. 1, which has two branches for the period 2-cycles, we can now deduce that part of the lower branch lying between the lines $x = 0$ and $x = 0.61541$ must be the locus of the point (a_{acc}, x_d), where the abscissa is the first accumulation point of the discontinuous map F with a sectional discontinuity at x_d. Thus, from Fig. 1 we can read off the values of a_{acc} of the map F for any value of x_d in the range $[0, 0.61541)$. For instance for the map

F with $x_d = 0$, the point $(1, 0)$ is on the lower branch implying that the first-level inverse cascade accumulates at $a = a_{acc} = 1$, a result which is in agreement with that found previously [5]. Note that we do not consider the rest of the lower branch and the upper branch of Fig. 1, as their ordinates fall outside the range [0, 0.61541).

From Fig. 1 we can deduce that as x_d increases from 0 until 0.61541, a_{acc} decreases from 1 to about 0.751. This agrees with the results shown in Table 2 that for positive values of x_d up to 0.6 the value of a_{acc} of F decreases monotonously with increasing x_d.

When x_d is negative, Table 1 shows that either the first-level inverse cascades do not exist or if they do, they do not occur immediately at the end of a 2-cycle, implying that the conclusion deduced above does not hold.

For $x_d \geq 0.61541$, both 2-cycle elements of F are located on the left-hand branch L. A part of the bifurcation diagram of the continuous map defined by

$$L'(x_n) = 0.9 - a\,x_n^2, \quad x_n \in [-1, 1] \tag{3}$$

is given in Fig. 3, showing a small portion of the fixed points, the 2-cycle elements as well as a portion of the 4-cycle elements. Note that the equations for the maps L and L' are the same but defined over different domains. The corresponding bifurcation diagram of the map F, with $x_d \geq 0.61541$ and $a \in [a_t, a_{acc})$, is identical to Fig. 3, where a_t is that value of a at which the fixed point of F jumps from the right branch R to the left branch L. An example of such a bifurcation diagram of F with $x_d = 0.8$ and $a \in [0, 1]$ is shown in Fig. 4 with a_t labelled.

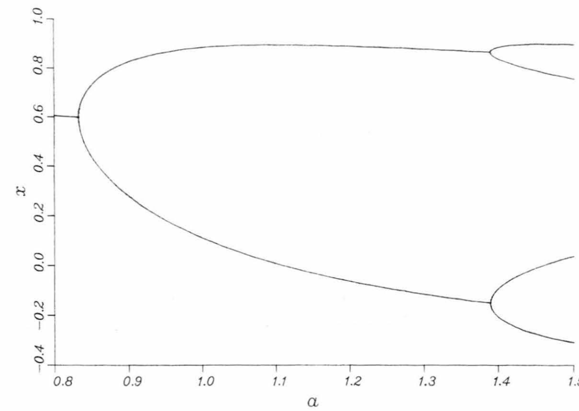

Fig. 3. Bifurcation diagram for the map L' defined by (3).

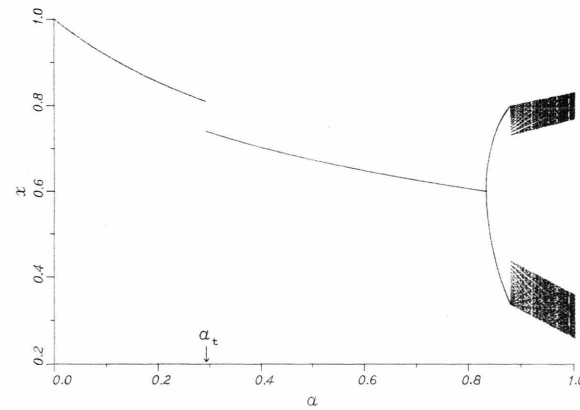

Fig. 4. Bifurcation diagram for the map F with $x_d = 0.8$. a_t is the value of a at which the fixed point of F jumps from the right branch R to the left branch L.

We can now use an argument similar to the one given earlier to deduce from Fig. 3 the value of a_{acc} for $x_d \geq 0.61541$. Thus, the upper branch for the period 2-cycles between $x_d = 0.61541$ to almost 0.9, beyond which there are no inverse cascades, must be the locus of the point (a_{acc}, x_d), where the abscissa is the first accumulation point of the discontinuous map F with a sectional discontinuity at x_d.

From the shape of the upper branch for the period 2 cycles shown in Fig. 3 we can easily understand why a_{acc} should increase as x_d increases from 0.65 to 0.8, as given in Table 2.

Note that for $x_d \geq 0.9$, though the 2-cycle elements fall on the left-hand branch L, we do not observe any inverse cascade at the end of the 2-cycle or anywhere else. Instead, there are cycles with periods 4, 8, 16, 32, 64 ... which lead to chaos by the usual period-doubling route. The absence of any inverse cascade is due to the fact that the discontinuity of the map F is located near its extreme right, and hence the discontinuous map behaves effectively as if it were continuous.

B) *Analytical expressions for a_{acc}*

For positive values of x_d of the map F we shall now derive analytical expressions for a_{acc}, the accumulation points of the first-level inverse cascades, whose numerical values are given in Table 2. As a consequence of the definition of the map F, which consists of a left branch $L(x)$ for $x \leq x_d$ and a right branch $R(x)$ for $x > x_d$, the derivations of the expressions for a_{acc}

fall into two cases. We shall make use of the following deductions: (i) When $x_d \in [0, 0.61541)$ and the parameter a is just smaller than a_{acc}, the 2-cycle elements of F fall on R, with one of them very close to x_d. Hence, when a of the map R' is equal to a_{acc} of the map F, one of the 2-cycle elements of R' is x_d. (ii) When $x_d \in [0.61541, 0.9)$ and the parameter a is just smaller than a_{acc}, the 2-cycle elements of F fall on L, with one of them at x_d. Hence when a of the map L' is equal to a_{acc} of the map F, one of the 2-cycle elements of L' is $x_d + \varepsilon$, where ε is an infinitesimal.

Case (a): $0 < x_d < 0.61541$.

Since when a of R' is equal to a_{acc} of F, one 2-cycle element of R' is equal to x_d, we have

$$R'^2(a_{acc}, x_d) = x_d$$

or

$$1 - x_d - a_{acc}(1 - a_{acc}\, x_d^2)^2 = 0. \tag{4}$$

By letting $b = 1 - a_{acc}\, x_d^2$, we have

$$a_{acc} = (1 - b)/x_d^2. \tag{5}$$

Equation (4) becomes

$$b^3 - b^2 - x_d^3 + x_d^2 = 0$$

with roots given by

$$b = x_d \text{ or}$$

$$b = \frac{-(x_d - 1) \pm \sqrt{(-3x_d^2 + 2x_d + 1)}}{2}.$$

As the first root corresponds to a fixed point of $R'(a_{acc})$ and b is positive, the only root of interest is

$$b = \frac{-(x_d - 1) + \sqrt{(-3x_d^2 + 2x_d + 1)}}{2}.$$

Hence, from (5) we get

$$a_{acc} = \frac{(1 + x_d) - \sqrt{(-3x_d^2 + 2x_d + 1)}}{2x_d^2}, \tag{6}$$

which is a general expression for the accumulation point a_{acc} of the first-level inverse cascade for any value of x_d in the range $(0, 0.61541)$. We can use this expression to obtain values of a_{acc} for the selected values of x_d in the range (0.01, 0.6) used previously [4]; these values of a_{acc}, shown in the third column of Table 2, agree numerically with those, shown in the second column, obtained previously by a computational technique for determining the periods of cycles.

Case (b): $0.61541 \le x_d < 0.9$.

Since when a of L' is equal to a_{acc} of F, one 2-cycle element of L' is equal to $x_d + \varepsilon$, we have

$$L'^2(a_{acc}, x_d + \varepsilon) = x_d + \varepsilon.$$

Since ε is an infinitesimal, the last equation can be replaced by

$$L'^2(a_{acc}, x_d) = x_d.$$

A derivation similar to the one in Case (a) leads to the following expression for a_{acc}:

$$a_{acc} = \frac{(9 + 10x_d) - \sqrt{(-300x_d^2 + 180x_d + 81)}}{20x_d^2}. \tag{7}$$

This general formula is valid for any value of x_d in the range (0.61541, 0.9); in particular, it yields the values of a_{acc} shown in the third column of Table 2 for $x_d = 0.65$, 0.7 and 0.8. These values again agree numerically with those, shown in the second column, obtained by direct computation [4].

It should be noted that the numerical values given in the second column of Table 2 are all rounded up to the next nearest five decimal places. Further, we have not provided analytical expressions for a_{acc} for $x_d = -0.3$, -0.2 or -0.01 since in these maps, the inverse cascades are not preceded by the 2-cycles but by the 4- or 8-cycles. Though we can generalize the technique given above to these maps with negative values of x_d, it is difficult to obtain an explicit expression for a_{acc} since the polynomial equations involved have degrees higher than quartic.

From the general analytical expressions for a_{acc} given in (6) and (7) we obtain the variation of a_{acc} with the position of the discontinuity x_d as shown in Figure 5. Note that there is a discontinuity in the graph at $x_d = 0.61541$.

III. Bifurcation Points

In our numerical study of the map F in (1) over a complete range of the parameter a we have determined the periods of the stable cycles which are classified into period-doubling sequences, inverse cascades and so on, as given in Table 1. Thus this map possesses many bifurcation points which are the values of the parameter a at which the cycle undergoes a change of periods. Here we shall only be concerned with bifurcation points within inverse cascades.

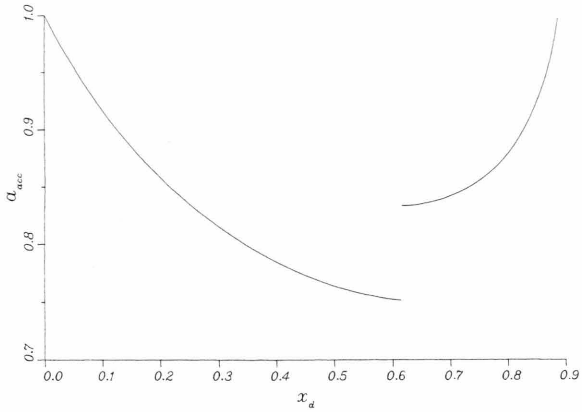

Fig. 5. Graph of accumulation point a_{acc} against the position of the discontinuity x_{d}.

We shall now use a technique that enables us to verify the previously-obtained values of the bifurcation points a_{bif} within any inverse cascade. This technique requires the knowledge of the exact order in which the R and L branches of the map F are visited after the iterates have converged to a stable cycle, i.e. of the order of cycling of the elements. Moreover, we must use the deduction that bifurcations within an inverse cascade occur whenever one of the cycle elements approaches the discontinuity of the map [4, 5]. We shall illustrate this technique by the following simple example.

Figure 6 shows a 6-cycle of the map F with $x_{\text{d}}=0.1$ at a parameter $a=0.981$. Five of the cycle elements lie on the right-hand branch R, with one of them very close to x_{d}, while the remaining single cycle element lies on the left-hand branch L. This figure depicts the situation shortly before the 6-cycle, a term of the first-level inverse cascade, bifurcates.

We shall now use the deduction that one of the cycle elements of this 6-cycle is located at $x_{\text{d}}+\varepsilon$, where ε is an infinitesimal. From the positions of the 6-cycle elements illustrated in Fig. 6, we arrive at the equation

$$\begin{aligned} x_{\text{d}} + \varepsilon &= F^6(x_{\text{d}} + \varepsilon) \\ &= R(R(R(L(R(R(x_{\text{d}} + \varepsilon)))))) = R^3 L R^2 (x_{\text{d}} + \varepsilon), \end{aligned}$$

which leads to

$$x_{\text{d}} = R^3 L R^2 (x_{\text{d}}) \tag{8}$$

since ε is an infinitesimal. By substituting $x_{\text{d}}=0.1$ and the expressions for R and L into (8), we get

$$0.1 = 1 - a(1 - a(1 - a(0.9 - a(1 - a(1 - 0.01\,a)^2)^2)^2)^2)^2$$

or

$$0.9 - a(1 - a(1 - a(0.9 - a(1 - a(1 - 0.01\,a)^2)^2)^2)^2)^2 = 0. \tag{9}$$

This equation can be solved numerically, and it is found that the value of one of the roots, which is given by $a=0.98128055665831524616519079880129$, is in excellent agreement to the last decimal place with the value of a_{bif} obtained previously by using a straightforward iterative method to determine the periods of cycles.

In order to compare the relevant root of the polynomial equation with that obtained by the direct computational method to the same number of significant figures, it is clearly necessary to use the same precision for both solving the polynomial equation and for determining a_{bif} directly. For example, if the same precision is used in the two methods, we are able to obtain the relevant root of (9) which is identical in value to that obtained by the direct computational method to the 32nd decimal place.

The above technique can be applied to verify values of a_{bif} of other cycles at any value of x_{d}. Thus, in this way we are able to verify the previously-determined values of a_{bif} by using a different approach. It follows that within any inverse cascade, the periods of cycles that were previously determined must be correct, no matter how long they may be. Of course, at the accumulation point of the inverse cascade, where the map

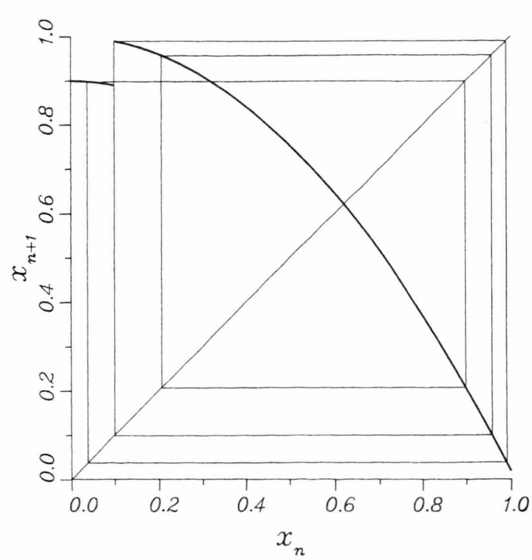

Fig. 6. The map F with $x_{\text{d}}=0.1$, showing a 6-cycle at $a=0.981$, shortly before bifurcation occurs.

has a relative, apparently real period (RARP) [11], the computer would not be able to determine the true period.

IV. Application to the TB Map

In this section, we shall show that the above technique for finding the value of a bifurcation point can be applied to the TB map which consists of two piecewise linear portions [10] defined by

$$G(x_n) = \begin{cases} L(x_n) \equiv 2x_n & \text{if } 0 \le x_n \le x_d, \\ R(x_n) \equiv C x_n + D & \text{if } x_d < x_n \le 1, \end{cases} \tag{10}$$

where $C = 4b - 2$, $D = 2 - 3b$, $x_d = 0.5$, and b is the control parameter.

This map has a large periodic region between two chaotic regions. Near the end of the periodic region there are many tiny periodic windows, in one of which each orbit has a period of eleven. At the commencement of this 11-cycle, one cycle element coincides with the position of the gap at $x_d = 0.5$. Using this information and the order of cycling of the elements, we obtain the following equation for the parameter of the bifurcation point:

$$x_d = G^{11}(x_d) = R L R^2 L R^2 L R^2 L(x_d) \tag{11}$$

which reduces to

$$C(2C(C(2C(C(2C(C+D) +2D)+D)+2D)+D)+2D)+D-0.5=0. \tag{12}$$

One of the roots is $b = 0.640531642310668437481985 05654816$, which agrees identically with that obtained directly by computing the periods of the orbits as a function of the parameter b.

V. Conclusions

We have seen that for an inverse cascade belonging to the map F in (1), there will always be a cycle element located at the discontinuity x_d of the map whenever bifurcation occurs and also when the cascade accumulates. This implies that a discontinuity in the map is probably a necessary condition for inverse cascades to arise.

We have derived two general analytical expressions for the accumulation points a_{acc} of F, one for x_d in the range (0, 0.61541) with the 2-cycle elements lying on the right-hand branch R for $a < a_{acc}$, and the other for $0.61541 \le x_d < 0.9$ with the 2-cycle elements lying on the left-hand branch L. The values of a_{acc} deduced from the analytical expressions for selected values of x_d are in excellent agreement with those previously obtained by a computational technique, thereby indicating the accuracy of the previous work [4].

By using two vastly different techniques on two different maps, one by direct computation [4, 5] and the other by determining the solution of a polynomial equation, we obtain identical values of the bifurcation points. This eliminates any doubt as to whether the cycles within an inverse cascade are real and the periods correctly determined. In fact, only at the accumulation point of each inverse cascade we obtain a fictitious period, as its true period is infinite but in practice, a finite period that is precision-dependent is obtained [11].

Acknowledgement

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